

ULTRAMETRIC DIFFUSION, EXPONENTIAL LANDSCAPES, AND THE FIRST PASSAGE TIME PROBLEM

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ABSTRACT. In this article we study certain ultradiffusion equations connected with energy landscapes of exponential type. These equations were introduced by Avetisov et al. in connection with certain p -adic models of complex systems, [5]-[4]. We show that the fundamental solutions of these equations are transition density functions of Lévy processes with state space \mathbb{Q}_p^n , we study some aspects of these processes including the first passage time problem.

1. INTRODUCTION

Stochastic processes on ultrametric spaces have received a lot attention in the latest years due to their connections with models of complex systems, see e.g. [2], [4]-[7], [12], [14]-[15], [16], [17], [20], [25]-[29], [33], [38]-[40], [42], [43], [44], and the references therein. A central paradigm in physics of complex systems (such proteins or glasses) asserts that the dynamics of such systems can be modeled as a random walk in the energy landscape of the system, see e.g. [21]-[23], [29], and the references therein. Typically these landscapes have a huge number of local minima. It is clear that a description of the dynamics on such landscapes require an adequate approximation. The interbasin kinetics offers an acceptable solution to this problem. By using this approach an energy landscape is approximated by an ultrametric space (a rooted tree) and a function on this space describing the distribution of the activation barriers, see e.g. [8], [34]-[35]. In this setting the dynamics of a complex system is codified in a master equation which describes the temporal behavior of the jumping probability between two states of the system, see [29]. In [5]-[6] Avetisov et al. introduced a new class of models for complex systems based on p -adic analysis, these models can be applied, for instance, to study the relaxation of biological complex systems. In this article we continue the study of these models, more precisely, we study n -dimensional versions of the master equations introduced in [5], see also [4], for exponential landscapes. We establish rigorously that such equations are ultradiffusion equations, i.e. we show that the fundamental solutions of these equations are transition density functions of Lévy processes with space state \mathbb{Q}_p^n , see Theorem 2. We also study the first passage time problem for the processes constructed in this article, see Theorem 3. Finally, we want to comment that it is not possible, due to physical and mathematical reasons, to forget the ultradiffusion equations and work exclusively with the attached Markovian semigroups. These semigroups have been extensively studied in the case of totally disconnected groups, see e.g. [9], [20]. For instance, in the study of the first passage

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time problem, the ultradiffusion equation itself plays a central role, see Lemmas 7-8. In a long term perspective, the goal is to extend the results presented here to equations of variable coefficients to obtain a theory similar to the one presented in [14]. A detailed discussion of our results (from the perspective of the Avetisov et al. models) and as well as a comparison with current literature is given in Section 3.

2. p -ADIC ANALYSIS: ESSENTIAL IDEAS

2.1. The field of p -adic numbers. Along this article p will denote a prime number. The field of p -adic numbers \mathbb{Q}_p is defined as the completion of the field of rational numbers \mathbb{Q} with respect to the p -adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0, & \text{if } x = 0 \\ p^{-\gamma}, & \text{if } x = p^\gamma \frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p . The integer $\gamma := \text{ord}(x)$, with $\text{ord}(0) := +\infty$, is called the p -adic order of x . We extend the p -adic norm to \mathbb{Q}_p^n by taking

$$\|x\|_p := \max_{1 \leq i \leq n} |x_i|_p, \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

We define $\text{ord}(x) = \min_{1 \leq i \leq n} \{\text{ord}(x_i)\}$, then $\|x\|_p = p^{-\text{ord}(x)}$. The metric space $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is a complete ultrametric space. As a topological space \mathbb{Q}_p is homeomorphic to a Cantor-like subset of the real line, see e.g. [1], [40].

Any p -adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{\text{ord}(x)} \sum_{j=0}^{\infty} x_j p^j,$$

where $x_j \in \{0, 1, 2, \dots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define the *fractional part* of $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0, & \text{if } x = 0 \text{ or } \text{ord}(x) \geq 0 \\ p^{\text{ord}(x)} \sum_{j=\text{ord}(x)}^{\infty} x_j p^j, & \text{if } \text{ord}(x) < 0. \end{cases}$$

In addition, any p -adic number $x \neq 0$ can be represented uniquely as $x = p^{\text{ord}(x)} ac(x)$ where $ac(x) = \sum_{j=0}^{\infty} x_j p^j$, $x_0 \neq 0$, is called the *angular component* of x . Notice that $|ac(x)|_p = 1$.

2.2. Additive characters. Set $\chi_p(y) = \exp(2\pi i \{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on \mathbb{Q}_p , i.e. a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of \mathbb{Q}_p form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$, the isomorphism is given by $\xi \rightarrow \chi_p(\xi x)$, see e.g. [1, Section 2.3].

2.3. Topology of \mathbb{Q}_p^n . For $r \in \mathbb{Z}$, denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n; \|x - a\|_p \leq p^r\}$ the ball of radius p^r with center at $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$, and take $B_r^n(0) := B_r^n$. Note that $B_r^n(a) = B_r(a_1) \times \dots \times B_r(a_n)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p; |x - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 = \mathbb{Z}_p$, the ring of p -adic integers. We also denote by $S_r^n(a) = \{x \in \mathbb{Q}_p^n; \|x - a\|_p = p^r\}$ the sphere of radius p^r with center at

$a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$, and take $S_r^n(0) := S_r^n$. We notice that $S_0^1 = \mathbb{Z}_p^\times$ (the group of units of \mathbb{Z}_p), but $(\mathbb{Z}_p^\times)^n \subsetneq S_0^n$. The balls and spheres are both open and closed subsets in \mathbb{Q}_p^n . In addition, two balls in \mathbb{Q}_p^n are either disjoint or one is contained in the other.

As a topological space $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is totally disconnected, i.e. the only connected subsets of \mathbb{Q}_p^n are the empty set and the points. A subset of \mathbb{Q}_p^n is compact if and only if it is closed and bounded in \mathbb{Q}_p^n , see e.g. [40, Section 1.3], or [1, Section 1.8]. The balls and spheres are compact subsets. Thus $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is a locally compact topological space.

We will use $\Omega(p^{-r}\|x - a\|_p)$ to denote the characteristic function of the ball $B_r^n(a)$. We will use the notation 1_A for the characteristic function of a set A .

2.4. The Bruhat-Schwartz space and the Fourier transform. A complex-valued function φ defined on \mathbb{Q}_p^n is called *locally constant* if for any $x \in \mathbb{Q}_p^n$ there exist an integer $l(x) \in \mathbb{Z}$ such that

$$(2.1) \quad \varphi(x + x') = \varphi(x) \text{ for } x' \in B_{l(x)}^n.$$

A function $\varphi : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ is called a *Bruhat-Schwartz function* (or a *test function*) if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The \mathbb{C} -vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^n)$. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, the largest number $l = l(\varphi)$ satisfying (2.1) is called *the exponent of local constancy* (or *the parameter of constancy*) of φ .

If U is an open subset of \mathbb{Q}_p^n , $\mathcal{D}(U)$ denotes the space of test functions with supports contained in U , then $\mathcal{D}(U)$ is dense in

$$L^\rho(U) = \left\{ \varphi : U \rightarrow \mathbb{C}; \|\varphi\|_\rho = \left\{ \int_{\mathbb{Q}_p^n} |\varphi(x)|^\rho d^n x \right\}^{\frac{1}{\rho}} < \infty \right\},$$

where $d^n x$ is the Haar measure on \mathbb{Q}_p^n normalized by the condition $\text{vol}(B_0^n) = 1$, for $1 \leq \rho < \infty$, see e.g. [1, Section 4.3].

2.5. The Fourier transform of test functions. Given $\xi = (\xi_1, \dots, \xi_n)$ and $y = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$, we set $\xi \cdot x := \sum_{j=1}^n \xi_j x_j$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x) \varphi(x) d^n x \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where $d^n x$ is the normalized Haar measure on \mathbb{Q}_p^n . The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^n)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$, see e.g. [1, Section 4.8]. We will also use the notation $\mathcal{F}_{x \rightarrow \xi} \varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ .

2.6. Distributions. Let $\mathcal{D}'(\mathbb{Q}_p^n)$ denote the \mathbb{C} -vector space of all continuous functionals (distributions) on $\mathcal{D}(\mathbb{Q}_p^n)$. The natural pairing $\mathcal{D}'(\mathbb{Q}_p^n) \times \mathcal{D}(\mathbb{Q}_p^n) \rightarrow \mathbb{C}$ is denoted as (T, φ) for $T \in \mathcal{D}'(\mathbb{Q}_p^n)$ and $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, see e.g. [1, Section 4.4].

Every $f \in \mathcal{E}(\mathbb{Q}_p^n)$, or more generally in L_{loc}^1 , defines a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^n)$ by the formula

$$(f, \varphi) = \int_{\mathbb{Q}_p^n} f(x) \varphi(x) d^n x.$$

Such distributions are called *regular distributions*.

2.7. The Fourier transform of a distribution. The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'(\mathbb{Q}_p^n)$ is defined by

$$(\mathcal{F}[T], \varphi) = (T, \mathcal{F}[\varphi]) \text{ for all } \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

The Fourier transform $T \rightarrow \mathcal{F}[T]$ is a linear (and continuous) isomorphism from $\mathcal{D}'(\mathbb{Q}_p^n)$ onto $\mathcal{D}'(\mathbb{Q}_p^n)$. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\xi)]$.

3. p -ADIC MODELS OF RELAXATION OF COMPLEX SYSTEMS: DISCUSSION OF THE RESULTS.

The dynamics of a large class of complex systems such as glasses and proteins is described by a random walk on a complex energy landscape, see e.g. [22], [23], [29, and the references therein], [41]. An energy landscape (or simply a landscape) is a continuous function $\mathbb{U} : X \rightarrow \mathbb{R}$ that assigns to each physical state of a system its energy. In many cases we can take X to be a subset of \mathbb{R}^N . The term complex landscape means that function \mathbb{U} has many local minima. In this case the method of *interbasin kinetics* is applied, in this approach, the study of a random walk on a complex landscape is based on a description of the kinetics generated by transitions between groups of states (basins). Minimal basins correspond to local minima of energy, and large basins have hierarchical structure. The transition rate between basins is determined by the Arrhenius factor, which depends on the energy barrier between these basins. Procedures for constructing hierarchies of basins kinetics from any energy landscapes have been studied extensively, see e.g. [8], [34], [35]. By using these methods, a complex landscape is approximated by a *disconnectivity graph* (a rooted tree) and by a function on the tree describing the distribution of the activation energies. The dynamics of the system is then encoded in a system of kinetic equations of the form:

$$(3.1) \quad \frac{\partial}{\partial t} u(i, t) = - \sum_j \{T(i, j)u(i, t) - T(j, i)u(j, t)\} v(j),$$

where the indices i, j number the states of the system (which correspond to local minima of energy), $T(i, j) \geq 0$ is the probability per unit time of a transition from i to j , and the $v(j) > 0$ are the basin volumes. For further details the reader may consult [29, and the references therein]. Several models of interbasin kinetics and hierarchical dynamics have been studied, see e.g. [25], [29, and the references therein], [31].

In [5]-[6] Avetisov et al. developed new class of models of interbasin kinetics using ultrametric diffusion generated by p -adic pseudodifferential operators. In these models, the time-evolution of the system is controlled by a master equation of the form

$$(3.2) \quad \frac{\partial u(x, t)}{\partial t} = \int_{\mathbb{Q}_p} \{j(x | y)u(y, t) - j(y | x)u(x, t)\} dy, \quad x \in \mathbb{Q}_p, \quad t \in \mathbb{R}_+,$$

where the function $u(x, t) : \mathbb{Q}_p \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a probability density distribution, and the function $j(x | y) : \mathbb{Q}_p \times \mathbb{Q}_p \rightarrow \mathbb{R}_+$ is the probability of transition from state y to the state x per unit time. Master equation (3.2) is a continuous version of (3.1) obtained from it by passing to a ‘continuous limit’ in \mathbb{Q}_p under the conditions

$v(j) = 1$, $T(i, j) = j(|i - j|_p)$, see e.g. [29]. The transition from a state y to a state x can be perceived as overcoming the energy barrier separating these states. In [5] an Arrhenius type relation was used, that is,

$$j(x | y) \sim A(T) \exp \left\{ -\frac{\mathbb{U}(x | y)}{kT} \right\},$$

where $\mathbb{U}(x | y)$ is the height of the activation barrier for the transition from the state y to state x , k is the Boltzmann constant and T is the temperature. This formula establishes a relation between the structure of the energy landscape $\mathbb{U}(x | y)$ and the transition function $j(x | y)$. The case $j(x | y) = j(y | x)$ corresponds to a *degenerate energy landscape*. In this case the master equation (3.2) takes the form

$$(3.3) \quad \frac{\partial u(x, t)}{\partial t} = \int_{\mathbb{Q}_p} j(|x - y|_p) \{u(y, t) - u(x, t)\} dy,$$

where $j(|x - y|_p) = \frac{A(T)}{|x - y|_p} \exp \left\{ -\frac{\mathbb{U}(|x - y|_p)}{kT} \right\}$. By choosing \mathbb{U} conveniently, several energy landscapes can be obtained. Following [5], there are three basic landscapes: (i) (logarithmic) $j(|x - y|_p) = \frac{1}{|x - y|_p \ln^\alpha(1 + |x - y|_p)}$, $\alpha > 1$ (ii) (linear) $j(|x - y|_p) = \frac{1}{|x - y|_p^{\alpha+1}}$, $\alpha > 0$, (iii) (exponential) $j(|x - y|_p) = \frac{e^{-\alpha|x - y|_p}}{|x - y|_p}$, $\alpha > 0$.

In our opinion, the novelty and relevance of the idealistic models of Avetisov et al. come from two facts: first, they codify, in a mathematical language, the central physical paradigm asserting that the dynamics of several complex systems can be described as a random walk on a rooted tree; second, these models give a description of the characteristic types of relaxation of complex systems.

The original models of Avetisov et al. were formulated in dimension one. The corresponding master equations were obtained by studying a random walk on a disconnectivity graph, which comes from ‘one’ cross section of the energy landscape of a system, by using the above mentioned limit process, the tree becomes in \mathbb{Q}_p . In [21] Frauenfelder et al. have explicitly pointed out that using rooted trees (disconnectivity graphs) constructed from ‘one’ cross section of an energy landscape of a complex systems is misleading in two respects: “it appears that a transition from an initial state i to a final state j must follow a unique pathway, and second entropy does not play a role,” see [21, p. 98 and figures 11.3 and 11.4]. If we consider several cross sections of an energy landscape, each of them gives rise to a disconnectivity graph, and hence the indices i, j in equation (3.2) are vectors running on the Cartesian product of the disconnectivity graphs; in the continuous model, see (3.3), the variables x, y run through \mathbb{Q}_p^n . Therefore, the master equations for the Avetisov et al. models should be studied in arbitrary dimension due to physical reasons and for mathematical generality. This program is being developed by the second author and his collaborators in recent years, see e.g. [12], [14], [15], [33], [38], [43], [44].

This article aims to study some aspects of the dynamics of ‘random walks’ associated with the exponential landscapes introduced in [5]. The corresponding master

equation takes the form

$$(3.4) \quad \begin{cases} \frac{\partial u(x,t)}{\partial t} = (J * u(\cdot, t))(x) - u(x, t), & x \in \mathbb{Q}_p^n, \quad t \geq 0 \quad (A) \\ u(x, 0) = u_0(x), & \quad \quad \quad (B) \end{cases}$$

where $J : \mathbb{Q}_p^n \rightarrow \mathbb{R}_+$ belongs to a family of functions, depending on several parameters, which codify the structure of an energy landscape. The family of landscapes studied here is an ‘integrable’ generalization of the one-dimensional exponential landscapes introduced in [5]. The first step is to establish that (3.4)-(A) is an ultradiffusion equation, i.e. that its fundamental solution is the transition density of a Markov process with space state \mathbb{Q}_p^n . It is important to mention here, that this fact has been established rigorously only for the linear landscapes, in this case (3.4)-(A) becomes a p -adic heat equation, and these equations and their Markov processes have been studied intensively lately, see [12], [15], [14], [28], [40], [44], [43]. For the exponential and logarithmic landscapes of [5], the function j is only locally integrable, and thus the natural domain of the operator in the right hand-side of (3.3) is not evident, and it is not given in [5], hence the fact that (3.3) is an ultradiffusion equation in the case of exponential and logarithm landscapes was not established in [5]. By imposing to the function J the condition of being integrable, the operator in the right hand-side of (3.4)-(A) becomes a linear bounded operator on L^ρ , $1 \leq \rho \leq \infty$, and in the case of exponential-type landscapes, we show that (3.4)-(A) is an ultradiffusion equation. In this article we show that the fundamental solution of (3.4)-(A) is the transition density of a Lévy process, see Theorem 2. It is worth to mention that the real counterpart of equation (3.4)-(A) has been studied intensively, in this setting the equation has been used to model diffusion processes, see e.g. [3]. In the preface of [3] the authors show that for certain Markov processes their density transition functions satisfy (3.4)-(A). In this article, we investigate the converse of this situation, in a p -adic setting.

We also study the first passage time problem for stochastic processes $\mathfrak{J}(t, \omega)$ whose transition density functions satisfy (3.4)-(A)-(B), with $u_0(x)$ equals to the characteristic function of \mathbb{Z}_p^n . More precisely, we study the random variable $\tau_{\mathbb{Z}_p^n}(\omega)$ defined as the smallest time in which a path of $\mathfrak{J}(t, \omega)$ returns to \mathbb{Z}_p^n . We show that every path of any of these processes is sure to return to \mathbb{Z}_p^n , see Theorem 3.

It is important to mention that our results do not cover all the exponential landscapes introduced here, this will require the study of equations of type (3.4)-(A)-(B) in a more general setting. Finally the results in [15] present an n -dimensional generalization of the linear energy landscapes of [5], this generalization was achieved by generalizing the p -adic heat equations. There are several important differences between this article and [15]. First, the operators (Laplacians) considered in [15] are unbounded operators densely defined in suitable subspaces the $L^2(\mathbb{Q}_p^n)$ while the operators studied here are bounded operators defined in $L^\rho(\mathbb{Q}_p^n)$, $1 \leq \rho \leq \infty$; second, the fundamental solutions in [15] are integrable functions of the position while here the fundamental solutions are distributions which makes the study of the corresponding stochastic processes more involved.

4. PRELIMINARY RESULTS

4.1. Exponential Landscapes. In this section we give several technical results for the functions J that codify the structure of the energy landscapes studied in this article.

Set $\mathbb{R}_+ := \{x \in \mathbb{R}; x \geq 0\}$. We fix a continuous function $J : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and take $J(x) = J(\|x\|_p)$ for $x \in \mathbb{Q}_p^n$, then $J(x)$ is a *radial function* on \mathbb{Q}_p^n . In addition, we assume that $\int_{\mathbb{Q}_p^n} J(\|x\|_p) d^n x = 1$.

Definition 1. We say that function $J(\|x\|_p)$ is of *exponential type* if there exist positive real constants A, B, C_1 , and a real constant $\gamma > -n$ such that

$$A \|x\|_p^\gamma e^{-C_1 \|x\|_p} \leq J(\|x\|_p) \leq B \|x\|_p^\gamma e^{-C_1 \|x\|_p}, \text{ for any } x \in \mathbb{Q}_p^n.$$

Remark 1. The condition $\gamma > -n$ is completely necessary to assure that $J \in L^1$. We notice that in dimension one the function $\frac{e^{-C|x|_p}}{|x|_p}$, $C > 0$, which was used in [5] as a j function, is not integrable. Indeed, assume that $\frac{e^{-C|x|_p}}{|x|_p} \in L^1$, then the following integral exists:

$$\begin{aligned} \int_{\mathbb{Z}_p} \frac{e^{-C|x|_p}}{|x|_p} dx &= \int_{p\mathbb{Z}_p} \frac{e^{-C|x|_p}}{|x|_p} dx + \int_{\mathbb{Z}_p^\times} \frac{e^{-C|x|_p}}{|x|_p} dx \\ (4.1) \quad &= \int_{\mathbb{Z}_p} \frac{e^{-Cp^{-1}|x|_p}}{|x|_p} dx + e^{-C} (1 - p^{-1}). \end{aligned}$$

Now, since $C|x|_p \geq Cp^{-1}|x|_p$, we have $\int_{\mathbb{Z}_p} \frac{e^{-C|x|_p}}{|x|_p} dx - \int_{\mathbb{Z}_p} \frac{e^{-Cp^{-1}|x|_p}}{|x|_p} dx \leq 0$, which contradicts (4.1). This situation causes several mathematical problems that we will discuss later on.

Lemma 1. With the above notation, the following assertions hold:

- (i) $\hat{J}(\xi)$ is a real-valued, radial (i.e. $\hat{J}(\xi) = \hat{J}(\|\xi\|_p)$), and continuous function, satisfying $|\hat{J}(\|\xi\|_p)| \leq 1$ and $\hat{J}(0) = 1$;
- (ii) for $\xi \in \mathbb{Q}_p^n \setminus \{0\}$,

$$1 - \hat{J}(\|\xi\|_p) = \|\xi\|_p^{-n} J(p \|\xi\|_p^{-1}) + p^n \|\xi\|_p^{-n} \sum_{l=0}^{\infty} p^{nl} J(p^{1+l} \|\xi\|_p^{-1});$$

- (iii) if $-n < \gamma < 0$, then

$$1 - \hat{J}(\|\xi\|_p) \leq B_1 \|\xi\|_p^{-n-\gamma} e^{-C_1 p \|\xi\|_p^{-1}} + B_2 \|\xi\|_p^{-\gamma} e^{-C_1 p \|\xi\|_p^{-1}},$$

for $\xi \in \mathbb{Q}_p^n \setminus \{0\}$, where B_1, B_2 are positive constants.

Proof. (i) The Fourier transform of an integrable radial function is a real-valued continuous radial function. Notice that $|\hat{J}(\|\xi\|_p)| \leq \int_{\mathbb{Q}_p^n} |\chi_p(\xi \cdot x)| J(x) d^n x = \int_{\mathbb{Q}_p^n} J(\|x\|_p) d^n x = 1$. Now, since J is integrable $\hat{J}(0) = \int_{\mathbb{Q}_p^n} J(\|x\|_p) d^n x = 1$.

- (ii) Take $\xi = p^{ord(\xi)} \xi_0$, with $\xi_0 = (\xi_0^{(1)}, \dots, \xi_0^{(n)})$, $\|\xi_0\|_p = 1$, and $ord(\xi) \in \mathbb{Z}$, then

$$\hat{J}(\|\xi\|_p) - 1 = \int_{\mathbb{Q}_p^n} J(\|x\|_p) \left\{ \chi_p(p^{ord(\xi)} \xi_0 \cdot x) - 1 \right\} d^n x.$$

By changing variables as $y_i = p^{ord(\xi)} \xi_0^{(i)} x_i$ for $i = 1, \dots, n$, with $d^n x = p^{ord(\xi)n} d^n y$, we have

$$\begin{aligned} \widehat{J}(\|\xi\|_p) - 1 &= p^{ord(\xi)n} \int_{\mathbb{Q}_p^n} J(p^{ord(\xi)} \|y\|_p) \left\{ \chi_p \left(\sum_{i=1}^n y_i \right) - 1 \right\} d^n y \\ &= p^{ord(\xi)n} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(p^{ord(\xi)} \|y\|_p) \left\{ \chi_p \left(\sum_{i=1}^n y_i \right) - 1 \right\} d^n y \\ &= p^{ord(\xi)n} \sum_{j=1}^{\infty} J(p^{j+ord(\xi)}) \int_{\|y\|_p = p^j} \left\{ \chi_p \left(\sum_{i=1}^n y_i \right) - 1 \right\} d^n y. \end{aligned}$$

By changing variables as $y = p^{-j} z$, $d^n y = p^{nj} d^n z$, we get

$$\begin{aligned} \widehat{J}(\|\xi\|_p) - 1 &= p^{ord(\xi)n} \sum_{j=1}^{\infty} p^{nj} J(p^{j+ord(\xi)}) \int_{S_0^n} \left\{ \chi_p \left(p^{-j} \sum_{i=1}^n z_i \right) - 1 \right\} d^n z \\ &= p^{ord(\xi)n} \sum_{j=1}^{\infty} p^{nj} J(p^{j+ord(\xi)}) \begin{cases} -p^{-n} & \text{if } j \leq 0 \\ -1 - p^{-n} & \text{if } j = 1 \\ -1 & \text{if } j \geq 2 \end{cases} \\ &= -(1 + p^{-n}) p^{ord(\xi)n+n} J(p^{1+ord(\xi)}) - p^{ord(\xi)n} \sum_{j=2}^{\infty} p^{nj} J(p^{j+ord(\xi)}) \\ &= -p^{ord(\xi)n} J(p^{1+ord(\xi)}) - p^{ord(\xi)n+n} \sum_{l=0}^{\infty} p^{nl} J(p^{1+l+ord(\xi)}). \end{aligned}$$

(iii) From (ii) and the fact that J is of exponential type, we get that

$$(4.2) \quad 1 - \widehat{J}(\|\xi\|_p) \leq B p^\gamma \|\xi\|_p^{-n-\gamma} e^{-C_1 p \|\xi\|_p^{-1}} + B p^{\gamma+n} \|\xi\|_p^{-n-\gamma} \sum_{l=0}^{\infty} p^{(n+\gamma)l} e^{-C_1 p^{1+l} \|\xi\|_p^{-1}}.$$

By using that $-n < \gamma < 0$,

$$\begin{aligned} (4.3) \quad & p^\gamma \sum_{l=0}^{\infty} p^{nl} p^{\gamma l} e^{-C_1 p^{1+l} \|\xi\|_p^{-1}} \leq \sum_{l=0}^{\infty} p^{nl} e^{-C_1 p^{1+l} \|\xi\|_p^{-1}} \\ &= e^{-C_1 p \|\xi\|_p^{-1}} \sum_{l=0}^{\infty} p^{nl} e^{-C_1 p \|\xi\|_p^{-1} (p^l - 1)} = e^{-C_1 p \|\xi\|_p^{-1}} \left\{ 1 + \sum_{l=1}^{\infty} p^{nl} e^{-C_1 p \|\xi\|_p^{-1} (p^l - 1)} \right\}. \end{aligned}$$

By using that $p^l - 1 \geq p^{l-1}$ for any positive integer,

$$\begin{aligned} e^{-C_1 p \|\xi\|_p^{-1}} \left\{ 1 + \sum_{l=1}^{\infty} p^{nl} e^{-C_1 p \|\xi\|_p^{-1} (p^l - 1)} \right\} &\leq e^{-C_1 p \|\xi\|_p^{-1}} \left\{ 1 + \sum_{l=1}^{\infty} p^{nl} e^{-C_1 p^l \|\xi\|_p^{-1}} \right\} \\ &= e^{-C_1 p \|\xi\|_p^{-1}} \left\{ 1 + \frac{1}{1 - p^{-n}} \int_{\|y\|_p > 1} e^{-C_1 \|y\|_p \|\xi\|_p^{-1}} d^n y \right\} \end{aligned}$$

(4.4)

$$\leq e^{-C_1 p \|\xi\|_p^{-1}} \left\{ 1 + \frac{1}{1 - p^{-n}} \int_{\mathbb{Q}_p^n} e^{-C_1 \|y\|_p \|\xi\|_p^{-1}} d^n y \right\} \leq e^{-C_1 p \|\xi\|_p^{-1}} \left\{ 1 + A_0 \|\xi\|_p^n \right\},$$

where we used the well-known estimation

$$\int_{\mathbb{Q}_p^n} e^{-\tau \|y\|_p} d^n y \leq C_0 \tau^{-n} \text{ for } \tau > 0.$$

Therefore from (4.2)-(4.4),

$$1 - \widehat{J}(\|\xi\|_p) \leq B_1 \|\xi\|_p^{-n-\gamma} e^{-C_1 p \|\xi\|_p^{-1}} + B_2 \|\xi\|_p^{-\gamma} e^{-C_1 p \|\xi\|_p^{-1}} \text{ for } \xi \neq 0.$$

□

Remark 2. We notice that the fact that J is of exponential type implies that $\text{supp } J(\|x\|_p) \not\subseteq \mathbb{Z}_p^n$. Then $1 - \widehat{J}(1) > 0$. Indeed, by Lemma 1-(ii),

$$\begin{aligned} 1 - \widehat{J}(1) &= J(p) + p^n \sum_{l=0}^{\infty} p^{nl} J(p^{1+l}) = J(p) + \sum_{k=1}^{\infty} p^{nk} J(p^k) \\ &= J(p) + \frac{1}{1 - p^{-n}} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(\|x\|_p) d^n x. \end{aligned}$$

If $1 - \widehat{J}(1) = 0$, then $J(p) = 0$ and $J(\|x\|_p) \equiv 0$ for $x \in \mathbb{Q}_p^n \setminus \mathbb{Z}_p^n$, i.e. $\text{supp } J(\|x\|_p) \subseteq \mathbb{Z}_p^n$.

Lemma 2. If $-n < \gamma < 0$ and J is of exponential type, then

$$\frac{\Omega(\|\xi\|_p)}{1 - \widehat{J}(\|\xi\|_p)} \notin L^1(\mathbb{Q}_p^n, d^n \xi).$$

Proof. By using Lemma 1-(iii),

$$\begin{aligned} \int_{\mathbb{Z}_p^n} \frac{d^n \xi}{1 - \widehat{J}(\|\xi\|_p)} &\geq \int_{\mathbb{Z}_p^n} \frac{d^n \xi}{B_1 \|\xi\|_p^{-n-\gamma} e^{-C_1 p \|\xi\|_p^{-1}} + B_2 \|\xi\|_p^{-\gamma} e^{-C_1 p \|\xi\|_p^{-1}}} \\ &= \int_{\mathbb{Z}_p^n} \frac{\|\xi\|_p^\gamma e^{C_1 p \|\xi\|_p^{-1}} d^n \xi}{B_1 \|\xi\|_p^{-n} + B_2} = \sum_{j=0}^{\infty} \frac{p^{-j\gamma} e^{C_1 p^{j+1}}}{B_1 p^{jn} + B_2} \int_{\|\xi\|_p = p^{-j}} d^n \xi \\ &= (1 - p^{-n}) \sum_{j=0}^{\infty} \frac{p^{-j(n+\gamma)} e^{C_1 p^{j+1}}}{B_1 p^{jn} + B_2} = \infty. \end{aligned}$$

□

Remark 3. (i) A function $f : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ is called *positive definite*, if

$$\sum_{i,j=1}^m f(x_i - x_j) \lambda_i \overline{\lambda_j} \geq 0$$

for all $m \in \mathbb{N} \setminus \{0\}$, $x_1, \dots, x_m \in \mathbb{Q}_p^n$ and $\lambda_1, \dots, \lambda_m \in \mathbb{C}$. By a direct calculation one verifies that $\widehat{J}(\|\xi\|_p)$ is a positive definite function.

(ii) A function $f : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ is called *negative definite*, if

$$\sum_{i,j=1}^m \left(f(x_i) + \overline{f(x_j)} - f(x_i - x_j) \right) \lambda_i \overline{\lambda_j} \geq 0$$

for all $m \in \mathbb{N} \setminus \{0\}$, $x_1, \dots, x_m \in \mathbb{Q}_p^n$, $\lambda_1, \dots, \lambda_m \in \mathbb{C}$. A theorem due to Schoenberg asserts that a function $f : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ is negative definite if and only if the following two conditions are satisfied: (1) $f(0) \geq 0$ and (2) the function $x \mapsto e^{-tf(x)}$ is positive definite for all $t > 0$, cf. [10, Theorem 7.8]. By using Corollary 7.7 in [10, Theorem 7.8], the function $\widehat{J}(0) - \widehat{J}(\|\xi\|_p) = 1 - \widehat{J}(\|\xi\|_p)$ is negative definite, by applying Schoenberg Theorem, the function $e^{t(\widehat{J}(\|\xi\|_p)-1)}$ is positive definite for all $t > 0$.

4.2. A class of nonlocal p -adic Operators. We define the operator $Af = J * f - f$ with J as in Section 4.1. Then, for any $1 \leq \rho \leq \infty$, $A : L^\rho \rightarrow L^\rho$ gives rise to a well-defined linear bounded operator. Indeed, by the Young inequality

$$\|Af\|_{L^\rho} \leq \|J * f\|_{L^\rho} + \|f\|_{L^\rho} \leq \|J\|_{L^1} \|f\|_{L^\rho} + \|f\|_{L^\rho} \leq 2\|f\|_{L^\rho}.$$

Proposition 1. Consider $A : L^2(\mathbb{Q}_p^n) \rightarrow L^2(\mathbb{Q}_p^n)$ given by

$$Af(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left((\widehat{J}(\|\xi\|_p) - 1) \mathcal{F}_{x \rightarrow \xi} f \right),$$

and the Cauchy problem :

$$(4.5) \quad \begin{cases} \frac{\partial u}{\partial t}(x, t) = Au(x, t), & t \in [0, \infty), x \in \mathbb{Q}_p^n \\ u(x, 0) = u_0(x) \in \mathcal{D}(\mathbb{Q}_p^n). \end{cases}$$

Then

$$u(x, t) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x) e^{(\widehat{J}(\|\xi\|_p)-1)t} \widehat{u_0}(\xi) d^n \xi$$

is a classical solution of (4.5). In addition, $u(\cdot, t)$ is a continuous function for any $t \geq 0$.

Proof. The result follows from the following assertions.

Claim 1. $u(x, \cdot) \in C^1([0, \infty))$ and

$$\frac{\partial u}{\partial t}(x, t) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x) (\widehat{J}(\|\xi\|_p) - 1) e^{(\widehat{J}(\|\xi\|_p)-1)t} \widehat{u_0}(\xi) d^n \xi$$

for $t \geq 0$, $x \in \mathbb{Q}_p^n$.

The formula follows from the fact that

$$\left| \chi_p(-\xi \cdot x) e^{(\widehat{J}(\|\xi\|_p)-1)t} \widehat{u_0}(\xi) \right| \leq |\widehat{u_0}(\xi)| \in L^1$$

and that

$$\left| \chi_p(-\xi \cdot x) (\widehat{J}(\|\xi\|_p) - 1) e^{(\widehat{J}(\|\xi\|_p)-1)t} \widehat{u_0}(\xi) \right| \leq 2 |\widehat{u_0}(\xi)| \in L^1,$$

cf. Lemma 1-(i), by applying the Dominated Convergence Theorem.

Claim 2.

$$Au(x, t) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x) (\widehat{J}(\|\xi\|_p) - 1) e^{(\widehat{J}(\|\xi\|_p) - 1)t} \widehat{u_0}(\xi) d^n \xi$$

for $t \in [0, \infty)$, $x \in \mathbb{Q}_p^n$.

The formula follows from the fact that $u(x, t) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(e^{(\widehat{J}(\|\xi\|_p) - 1)t} \widehat{u_0}(\xi) \right) \in L^2(\mathbb{Q}_p^n)$ for any $t \geq 0$ and that $\left(\widehat{J}(\|\xi\|_p) - 1 \right) e^{(\widehat{J}(\|\xi\|_p) - 1)t} \widehat{u_0}(\xi) \in L^2(\mathbb{Q}_p^n)$ for any $t \geq 0$, cf. Lemma 1-(i). \square

5. HEAT KERNELS

We define the *heat Kernel* attached to operator A as

$$Z(x, t) = \mathcal{F}_{\xi \rightarrow x}^{-1} (e^{(\widehat{J}(\|\xi\|_p) - 1)t}) \in \mathcal{D}'(\mathbb{Q}_p^n) \text{ for } t \geq 0.$$

When considering $Z(x, t)$ as a function of x for t fixed we will write $Z_t(x)$.

We recall that a distribution $F \in \mathcal{D}'(\mathbb{Q}_p^n)$ is called *positive*, if $(F, \varphi) \geq 0$ for every *positive test function* φ , i.e. for $\varphi : \mathbb{Q}_p^n \rightarrow \mathbb{R}_+$, $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$.

A distribution F is *positive definite*, if for every test function φ , the inequality $(F, \overline{\varphi * \widetilde{\varphi}}) \geq 0$ holds, where $\widetilde{\varphi}(x) = \overline{\varphi(-x)}$ and $\overline{\varphi(-x)}$ denotes the complex conjugate of $\varphi(-x)$.

Theorem 1 (p -adic Bochner-Schwartz Theorem [44, Theorem 4.1]). *Every positive-definite distribution F on \mathbb{Q}_p^n is the Fourier transform of a regular Borel measure μ on \mathbb{Q}_p^n , i.e.*

$$(F, \varphi) = \int_{\mathbb{Q}_p^n} \widehat{\varphi}(\xi) d\mu(\xi), \text{ for } \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

Remark 4. *If $f : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ is a continuous positive-definite function, then*

$$(f, \overline{\varphi * \widetilde{\varphi}}) \geq 0 \text{ for } \varphi \in \mathcal{D}(\mathbb{Q}_p^n),$$

which means that f generates a positive-definite distribution, see e.g. [10, Proposition 4.1].

Lemma 3. *Let φ be a positive test function. Then*

$$\int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x) e^{(\widehat{J}(\|\xi\|_p) - 1)t} \widehat{\varphi}(\xi) d^n \xi = (Z_t * \varphi)(x) \geq 0 \text{ for } x \in \mathbb{Q}_p^n \text{ and } t \geq 0.$$

Proof. It is sufficient to show the lemma for $x \in \mathbb{Q}_p^n$ and $t > 0$. By Remark 3-(ii), the function $e^{t(\widehat{J}(\|\xi\|_p) - 1)}$ is positive definite for all $t > 0$, by Remark 4 and Theorem 1, $Z_t(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} (e^{(\widehat{J}(\|\xi\|_p) - 1)t})$ is a Borel measure on \mathbb{Q}_p^n for $t > 0$, which is identified with a positive distribution. Then

$$\begin{aligned} (e^{(\widehat{J}(\|\xi\|_p) - 1)t}, \chi_p(-\xi \cdot x) \widehat{\varphi}(\xi)) &= \left(\mathcal{F}_{\xi \rightarrow y}^{-1} (e^{(\widehat{J}(\|\xi\|_p) - 1)t}), \mathcal{F}_{\xi \rightarrow y} (\chi_p(-\xi \cdot x) \widehat{\varphi}(\xi)) \right) \\ &= (Z_t(y), \varphi(x - y)) \geq 0 \text{ for } t > 0, \end{aligned}$$

since $\varphi(x - y) \geq 0$. \square

5.1. Decaying of the heat kernel at infinity. Let $h\left(\|\xi\|_p\right) \in L_{\text{loc}}^1$, then

$$\sum_{j=-m}^m h(p^j) 1_{S_j^n}(\xi) \rightarrow h\left(\|\xi\|_p\right) \text{ in } \mathcal{D}'(\mathbb{Q}_p^n).$$

Now, by using [1, Theorem 4.9.3] and the fact that \mathcal{F}^{-1} is continuous on $\mathcal{D}'(\mathbb{Q}_p^n)$,

$$\begin{aligned} \mathcal{F}_{\xi \rightarrow x}^{-1} \left(\sum_{j=-m}^m h(p^j) 1_{S_j^n}(\xi) \right) &= \sum_{j=-m}^m h(p^j) \mathcal{F}_{\xi \rightarrow x}^{-1} \left(1_{S_j^n}(\xi) \right) \\ &= \sum_{j=-m}^m h(p^j) \int_{S_j^n} \chi_p(-x \cdot \xi) d^n \xi \text{ in } \mathcal{D}'(\mathbb{Q}_p^n), \end{aligned}$$

therefore

$$(5.1) \quad \mathcal{F}_{\xi \rightarrow x}^{-1} \left(h\left(\|\xi\|_p\right) \right) = \sum_{j=-\infty}^{\infty} h(p^j) \int_{S_j^n} \chi_p(-x \cdot \xi) d^n \xi \text{ in } \mathcal{D}'(\mathbb{Q}_p^n).$$

Suppose now that $\sum_{k=0}^{\infty} p^{-kn} h\left(p^{-k} \|x\|_p^{-1}\right) < \infty$, then

$$\begin{aligned} \tilde{h}(x) &:= \int_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) h\left(\|\xi\|_p\right) d^n \xi := \lim_{m \rightarrow \infty} \sum_{j=-m}^m \int_{S_j^n} \chi_p(-x \cdot \xi) h\left(\|\xi\|_p\right) d^n \xi \\ &= \sum_{j=-\infty}^{\infty} h(p^j) \int_{S_j^n} \chi_p(-x \cdot \xi) d^n \xi \\ &= (1 - p^{-n}) \|x\|_p^{-n} \sum_{k=0}^{\infty} p^{-kn} h\left(p^{-k} \|x\|_p^{-1}\right) - \|x\|_p^{-n} h\left(p \|x\|_p^{-1}\right) \text{ for } x \neq 0, \end{aligned}$$

and by comparing with (5.1), we get

$$\left(\mathcal{F}_{\xi \rightarrow x}^{-1} \left[h\left(\|\xi\|_p\right) \right], \phi(x) \right) = \left(\tilde{h}(x), \phi(x) \right)$$

for $\phi \in \mathcal{D}(\mathbb{Q}_p^n)$ with $\text{supp } \phi \subset \mathbb{Q}_p^n \setminus \{0\}$. It is important to highlight that function $\tilde{h}(x)$ is not the inverse Fourier transform of $h\left(\|\xi\|_p\right)$ in the classical sense, because we use an improper integral in its definition. This is the reason for the ‘new notation’. We formally summarize the above reasoning in the following lemma:

Lemma 4. *Let $h\left(\|\xi\|_p\right) \in L_{\text{loc}}^1$ satisfying $\sum_{k=0}^{\infty} p^{-kn} h\left(p^{-k} \|\xi\|_p^{-1}\right) < \infty$ for $\xi \neq 0$, then $\mathcal{F}_{\xi \rightarrow x}^{-1} \left[h\left(\|\xi\|_p\right) \right] = \tilde{h}(x)$ as a distribution on $\mathcal{D}(\mathbb{Q}_p^n \setminus \{0\})$.*

We now apply this lemma to the case $h\left(\|\xi\|_p\right) = e^{(\hat{J}(\|\xi\|_p) - 1)t}$, with $t \geq 0$:

$$(Z(x, t), \phi(x)) = (\tilde{Z}(x, t), \phi(x))$$

for $\phi \in \mathcal{D}(\mathbb{Q}_p^n)$ with $\text{supp } \phi \subset \mathbb{Q}_p^n \setminus \{0\}$, where

$$(5.2) \quad \tilde{Z}(x, t) = \|x\|_p^{-n} \left[(1 - p^{-n}) \sum_{j=0}^{\infty} p^{-nj} e^{(\hat{J}(p^{-j} \|x\|_p^{-1}) - 1)t} - e^{(\hat{J}(p \|x\|_p^{-1}) - 1)t} \right]$$

for $t \geq 0$ and $x \neq 0$.

Proposition 2. *Assume that J is of exponential type, then the following estimations hold:*

- (i) $\tilde{Z}(x, t) \leq 2t\|x\|_p^{-n}$, for $x \in \mathbb{Q}_p^n \setminus \{0\}$ and $t > 0$;
- (ii) if $-n < \gamma < 0$, then $\tilde{Z}(x, t) \leq C_0 t \|x\|_p^\gamma e^{-C_1 \|x\|_p}$, for $\|x\|_p > p^l$, $l \in \mathbb{Z}$, and $t > 0$, where the positive constant C_0 depends on l .

Proof. The estimations follow from the following Claim:

Claim. $\tilde{Z}(x, t) \leq t\|x\|_p^{-n} \left\{ 1 - \hat{J}(p\|x\|_p^{-1}) \right\}$, for $x \in \mathbb{Q}_p^n \setminus \{0\}$ and $t > 0$.

We notice that by Lemma 1 (ii) $1 - \hat{J}(p\|x\|_p^{-1}) \geq 0$. The first estimation follows from Lemma 1-(i) and the Claim. The second estimation follows from the Claim and Lemma 1-(iii).

The proof of the Claim is as follows. By using that $e^{(\hat{J}(p^{-j}\|x\|_p^{-1})-1)t} \leq 1$ for $j \in \mathbb{N}$, cf. Lemma 1-(i), we get that

$$\begin{aligned} \tilde{Z}(x, t) &\leq \|x\|_p^{-n} \left[(1 - p^{-n}) \sum_{j \geq 0} p^{-nj} - e^{(\hat{J}(p\|x\|_p^{-1})-1)t} \right] \\ &= \|x\|_p^{-n} \left[\sum_{j \geq 0} (p^{-nj} - p^{-n(j+1)}) - e^{(\hat{J}(p\|x\|_p^{-1})-1)t} \right] \\ &= \|x\|_p^{-n} \left\{ 1 - e^{(\hat{J}(p\|x\|_p^{-1})-1)t} \right\}. \end{aligned}$$

We now apply the Mean-Value Theorem to the real function $e^{(\hat{J}(p\|x\|_p^{-1})-1)u}$ on $[0, t]$ with $t > 0$,

$$e^{(\hat{J}(p\|x\|_p^{-1})-1)t} - 1 = \left\{ \hat{J}(p\|x\|_p^{-1}) - 1 \right\} t e^{(\hat{J}(p\|x\|_p^{-1})-1)\tau}$$

for some $\tau \in (0, t)$, consequently $1 - e^{(\hat{J}(p\|x\|_p^{-1})-1)t} \leq \left\{ 1 - \hat{J}(p\|x\|_p^{-1}) \right\} t$. Hence,

$$\tilde{Z}(x, t) \leq t\|x\|_p^{-n} \left\{ 1 - \hat{J}(p\|x\|_p^{-1}) \right\}.$$

□

6. LÉVY PROCESSES

For the basic results on Hunt, Lévy and Markov processes the reader may consult [18], [37], [11], [20].

Remark 5. We denote by \mathcal{B}_0 the family of subsets of \mathbb{Q}_p^n formed by finite unions of disjoint balls and the empty set. This family has a natural structure of Boolean ring, i.e. if $B_1, B_2 \in \mathcal{B}_0$ then $B_1 \cup B_2 \in \mathcal{B}_0$ and $B_1 \setminus B_2 \in \mathcal{B}_0$. The Caratheodory Theorem asserts that if μ is a σ -finite measure on \mathcal{B}_0 , then there is a unique measure also denoted by μ on $\mathcal{B}(\mathbb{Q}_p^n)$, the σ -ring generated by \mathcal{B}_0 , which is σ -ring of Borel sets of \mathbb{Q}_p^n , see [24, Theorem A, p. 54]. Then every positive distribution can be identified with a Borel measure on \mathbb{Q}_p^n .

Definition 2. For $E \in \mathcal{B}_0(\mathbb{Q}_p^n)$, we define

$$p_t(x, E) = \begin{cases} Z_t(x) * 1_E(x), & \text{for } t > 0, x \in \mathbb{Q}_p^n \\ 1_E(x), & \text{for } t = 0, x \in \mathbb{Q}_p^n. \end{cases}$$

Lemma 5. $p_t(x, \cdot)$, $t \geq 0$, $x \in \mathbb{Q}_p^n$, is a measure on $\mathcal{B}(\mathbb{Q}_p^n)$.

Proof. By Lemma 3 and the fact that

$$p_t(x, E) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x) e^{(\hat{J}(\|\xi\|_p) - 1)t} \widehat{1_E}(\xi) d^n \xi \text{ for } t > 0, x \in \mathbb{Q}_p^n$$

$p_t(x, E)$ is a measure on \mathcal{B}_0 , in addition, $p_t(x, \cdot)$ has a unique extension to a measure on the Borel σ -ring of \mathbb{Q}_p^n . We denote this extension also by $p_t(x, \cdot)$. Indeed, by the Caratheodory Theorem, see Remark 5, it is sufficient to show that \mathbb{Q}_p^n is a countable disjoint union of balls B_i^n , $i \in \mathbb{N}$, satisfying $p_t(x, B_i^n) < \infty$ for any $i \in \mathbb{N}$. Indeed,

$$\mathbb{Q}_p^n = \bigsqcup_{\tilde{x}_i \in (\mathbb{Q}_p/\mathbb{Z}_p)^n} B_0^n(\tilde{x}_i),$$

where the elements of $\mathbb{Q}_p/\mathbb{Z}_p$ have the form $\tilde{y} = a_{-m}p^{-m} + \dots + a_{-1}p^{-1}$ with $a_i \in \{0, \dots, p-1\}$. The correspondence $\tilde{y} \mapsto a_m p^m + \dots + a_1 p^1$ implies that $\mathbb{Q}_p/\mathbb{Z}_p$ is countable, and therefore $(\mathbb{Q}_p/\mathbb{Z}_p)^n$ is also countable. Finally, the condition $p_t(x, B_0^n(\tilde{x}_i)) < \infty$ follows from the fact that $\widehat{1_{B_0^n(\tilde{x}_i)}}(\xi)$ has compact support. \square

Proposition 3. $p_t(x, E)$ for $t \geq 0$, $x \in \mathbb{Q}_p^n$, $E \in \mathcal{B}(\mathbb{Q}_p^n)$ is a Markov transition function on \mathbb{Q}_p^n .

Proof. **Claim 1.** $p_t(x, \cdot)$ is measure on $\mathcal{B}(\mathbb{Q}_p^n)$ and $p_t(x, \mathbb{Q}_p^n) = 1$ for all $t \geq 0$ and $x \in \mathbb{Q}_p^n$.

The first part of the assertion was established in Lemma 5. To show the second part of the assertion, we notice that B_k^n , $k \in \mathbb{N}$, is an increasing sequence of Borelian sets converging to \mathbb{Q}_p^n , i.e. $B_k^n \uparrow \mathbb{Q}_p^n$. Set $\Delta_k(x) := \Omega(p^{-k}\|x\|_p)$, $k \in \mathbb{N}$, hence $p_t(x, \mathbb{Q}_p^n) = \lim_{k \rightarrow \infty} p_t(x, \Delta_k)$. Now, since

$$\widehat{\Delta_k}(\xi) = \delta_k(\xi) = p^{kn} \begin{cases} 1 & \text{if } \|\xi\|_p \leq p^{-k} \\ 0 & \text{if } \|\xi\|_p > p^{-k}, \end{cases}$$

$$\begin{aligned} p_t(x, \mathbb{Q}_p^n) &= \lim_{k \rightarrow \infty} \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x) e^{(\hat{J}(\|\xi\|_p) - 1)t} \delta_k(\xi) d^n \xi \\ &= \lim_{k \rightarrow \infty} p^{nk} \int_{\|\xi\|_p \leq p^{-k}} \chi_p(-\xi \cdot x) e^{(\hat{J}(\|\xi\|_p) - 1)t} d^n \xi \quad (\text{taking } p^{-k}\xi = z) \\ &= \lim_{k \rightarrow \infty} \int_{\|z\|_p \leq 1} \chi_p(-p^k z \cdot x) e^{(\hat{J}(p^{-k}\|z\|_p) - 1)t} d^n z \\ &= \lim_{k \rightarrow \infty} \int_{\|z\|_p \leq 1} e^{(\hat{J}(p^{-k}\|z\|_p) - 1)t} d^n z, \text{ for } k \text{ big enough,} \end{aligned}$$

because for k big enough $p^k x \in \mathbb{Z}_p^n$ and thus $\chi_p(-p^k z \cdot x) \equiv 1$. By using that $e^{(\hat{J}(p^{-k}\|z\|_p) - 1)t} \leq 1$ for any $t \geq 0$, and that $\lim_{k \rightarrow \infty} e^{(\hat{J}(p^{-k}\|z\|_p) - 1)t} = 1$ (\hat{J} is continuous at the origin and $\hat{J}(0) = 1$), and by applying the Dominated Convergence

Theorem,

$$p_t(x, \mathbb{Q}_p^n) = \int_{\|z\|_p \leq 1} d^n z = 1.$$

Claim 2. $p_t(\cdot, E)$ is a Borel measurable function for all $t > 0$ and $E \in \mathcal{B}(\mathbb{Q}_p^n)$.

Define $E_k = \Delta_k E$, $k \in \mathbb{N}$, then $E_k \uparrow E$ with $E_k \in \mathcal{B}(\mathbb{Q}_p^n)$. By abuse of language, we use the notation $p_t(x, E_k)$ to mean a function of (t, x) with E_k fixed. Now, $p_t(x, E_k)$ is the solution of

$$\begin{cases} \frac{\partial}{\partial t} p_t(x, E_k) = J(x) * p_t(x, E_k) - p_t(x, E_k), & t \in [0, \infty), x \in \mathbb{Q}_p^n \\ p_0(x, E_k) = 1_{E_k}, & 1_{E_k} \in L^1(\mathbb{Q}_p^n), \end{cases}$$

cf. Proposition 1. Then, $p_t(x, E_k)$ is a continuous function in x for any $t \geq 0$, which implies that $p_t(\cdot, E_k)$ is a measurable function of x for any $t \geq 0$. Now, by using that $E_k \uparrow E$ and the fact $p_t(x, \cdot)$ is a measure on $\mathcal{B}(\mathbb{Q}_p^n)$, we get that $p_t(x, E_k) \rightarrow p_t(x, E)$ as $k \rightarrow \infty$, which implies that $p_t(\cdot, E)$ is the pointwise limit of a sequence of measurable functions $\{p_t(\cdot, E_k)\}_k$ and consequently $p_t(\cdot, E)$ is measurable.

Claim 3. $p_0(x, \{x\}) = 1$ for all $x \in \mathbb{Q}_p^n$.

This is a direct consequence of the definition of measure $p_t(x, E)$.

Claim 4. (The Chapman-Kolmogorov equation) For all $t, s \geq 0$, $x \in \mathbb{Q}_p^n$ and $E \in \mathcal{B}(\mathbb{Q}_p^n)$,

$$p_{t+s}(x, E) = \int_{\mathbb{Q}_p^n} p_t(x, d^n y) p_s(y, E).$$

We consider the case $t, s > 0$, since in the other cases the assertion is clear. We first note that

$$(6.1) \quad p_{t+s}(x, \cdot) = p_t(x, \cdot) * p_s(x, \cdot) \text{ in } \mathcal{D}'(\mathbb{Q}_p^n).$$

Indeed, for $E \in \mathcal{B}_0$,

$$\begin{aligned} p_{t+s}(x, E) &= \mathcal{F}_{\xi \rightarrow x}^{-1} (e^{(\hat{J}(\|\xi\|_p) - 1)(t+s)}) * 1_E = \mathcal{F}_{\xi \rightarrow x}^{-1} (e^{(\hat{J}(\|\xi\|_p) - 1)t} e^{(\hat{J}(\|\xi\|_p) - 1)s}) * 1_E \\ &= \left[\mathcal{F}_{\xi \rightarrow x}^{-1} (e^{(\hat{J}(\|\xi\|_p) - 1)t}) * \mathcal{F}_{\xi \rightarrow x}^{-1} (e^{(\hat{J}(\|\xi\|_p) - 1)s}) \right] * 1_E, \end{aligned}$$

since $e^{(\hat{J}(\|\xi\|_p) - 1)t}, e^{(\hat{J}(\|\xi\|_p) - 1)s} \in L_{loc}^1$. The Chapman-Kolmogorov equation is exactly (6.1). Indeed, by using the fact that the convolution of distributions is associative, we get from (6.1) that

$$\begin{aligned} p_{t+s}(x, E) &= \mathcal{F}_{\xi \rightarrow x}^{-1} (e^{(\hat{J}(\|\xi\|_p) - 1)t}) * (\mathcal{F}_{\xi \rightarrow x}^{-1} (e^{(\hat{J}(\|\xi\|_p) - 1)s}) * 1_E) \\ (6.2) \quad &= \mathcal{F}_{\xi \rightarrow x}^{-1} (e^{(\hat{J}(\|\xi\|_p) - 1)t}) * p_s(x, E), \end{aligned}$$

$E \in \mathcal{B}_0$. We now recall that the convolution of a distribution and a test function is a locally constant function, and hence its Fourier transform, as a distribution, is a function with compact support. By using this, from (6.2), we have

$$\begin{aligned} p_{t+s}(x, E) &= \mathcal{F}_{\xi \rightarrow x}^{-1} \left(e^{(\hat{J}(\|\xi\|_p) - 1)t} \hat{p}_s(\xi, E) \right) \\ &= \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x) e^{(\hat{J}(\|\xi\|_p) - 1)t} \hat{p}_s(\xi, E) d^n \xi \end{aligned}$$

in $\mathcal{D}'(\mathbb{Q}_p^n)$. Let B_N^n be a ball containing the support of $\widehat{p}_s(\xi, E)$, with N depending on E and s , by using Fubini's Theorem,

$$\begin{aligned}
 p_{t+s}(x, E) &= \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x) \left(e^{(\widehat{J}(\|\xi\|_p)-1)t} 1_{B_N^n}(\xi) \right) \widehat{p}_s(\xi, E) d^n \xi \\
 &= \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x) \left(e^{(\widehat{J}(\|\xi\|_p)-1)t} 1_{B_N^n}(\xi) \right) \left\{ \int_{\mathbb{Q}_p^n} \chi_p(y \cdot \xi) p_s(y, E) d^n y \right\} d^n \xi \\
 &= \int_{\mathbb{Q}_p^n} \left(\int_{\mathbb{Q}_p^n} \chi_p(-(x-y) \cdot \xi) e^{(J(\|\xi\|_p)-1)t} 1_{B_N^n}(\xi) d^n \xi \right) p_s(y, E) d^n y \\
 (6.3) \quad &= \int_{\mathbb{Q}_p^n} p_t(x-y, \widehat{1}_{B_N^n}) p_s(y, E) d^n y \text{ for } E \in \mathcal{B}_0.
 \end{aligned}$$

Formula (6.3) between positive distributions extends to a formula between Borel measures on \mathbb{Q}_p^n , by the Caratheodory Theorem. \square

Remark 6. (i) The transition function $p_t(x, \cdot)$ is normal, i.e. $\lim_{t \rightarrow 0^+} p_t(x, \mathbb{Q}_p^n) = 1$ for all $x \in \mathbb{Q}_p^n$. This follows from the fact that $p_t(x, \mathbb{Q}_p^n) = 1$, see proof of Claim 1.

(ii) From (6.1) we have $\{p_t(x, \cdot)\}_{t \geq 0}$ is an convolution semigroup in $\mathcal{D}'(\mathbb{Q}_p^n)$, and moreover $p_t(x, \cdot) \rightarrow \delta$ when $t \rightarrow 0^+$.

(iii) A function $p(x, E)$, $x \in \mathbb{Q}_p^n$, $E \in \mathcal{B}(\mathbb{Q}_p^n)$, is called a sub-Markovian transition function on $(\mathbb{Q}_p^n, \mathcal{B}(\mathbb{Q}_p^n))$, if it satisfies (1) for every $x \in \mathbb{Q}_p^n$, $p(x, \cdot)$ is a measure on $\mathcal{B}(\mathbb{Q}_p^n)$ such that $p(x, \mathbb{Q}_p^n) \leq 1$; (2) for every $E \in \mathcal{B}(\mathbb{Q}_p^n)$, $p(\cdot, E)$ is a Borel measurable function. Therefore, $p_t(x, E)$ for $t \geq 0$, $x \in \mathbb{Q}_p^n$, $E \in \mathcal{B}(\mathbb{Q}_p^n)$ is a sub-Markov transition function on \mathbb{Q}_p^n .

Let $C_b(\mathbb{Q}_p^n)$ be the space of real-valued, bounded, and continuous functions on \mathbb{Q}_p^n . This is a Banach space with the norm $\|f\|_\infty = \sup_{x \in \mathbb{Q}_p^n} |f(x)|$. We say that a function $f \in C_b(\mathbb{Q}_p^n)$ converges to zero as $x \rightarrow \infty$ if, for each $\epsilon > 0$, there exists a compact subset $E \subset \mathbb{Q}_p^n$ such that $|f(x)| < \epsilon$ for all $x \in \mathbb{Q}_p^n \setminus E$. In such case we write $\lim_{x \rightarrow \infty} f(x) = 0$. We set

$$C_0(\mathbb{Q}_p^n) := \{f \in C_b(\mathbb{Q}_p^n); \lim_{x \rightarrow \infty} f(x) = 0\}.$$

The space $C_0(\mathbb{Q}_p^n)$ is a closed subspace of $C_b(\mathbb{Q}_p^n)$, and thus it is a Banach space.

Definition 3. Given a Markov transition function $p_t(x, \cdot)$, we attach to it the following operator:

$$T_t f(x) := \begin{cases} \int_{\mathbb{Q}_p^n} p_t(x, d^n y) f(y) & \text{if } t > 0 \\ f & \text{if } t = 0. \end{cases}$$

We say that $p_t(x, \cdot)$ is a C_0 -function if the space $C_0(\mathbb{Q}_p^n)$ is an invariant subspace for the operators T_t , $t \geq 0$, i.e.

$$f \in C_0(\mathbb{Q}_p^n) \longrightarrow T_t f \in C_0(\mathbb{Q}_p^n).$$

Lemma 6. $p_t(x, \cdot)$ is a C_0 -function. Furthermore, $T_t : C_0(\mathbb{Q}_p^n) \rightarrow C_0(\mathbb{Q}_p^n)$ is a bounded linear operator.

Proof. The result follows from the fact that $\mathcal{D}(\mathbb{Q}_p^n)$ is dense in $C_0(\mathbb{Q}_p^n)$, see e.g. [36, Proposition 1.3], by the following Claim:

Claim. $T_t : (\mathcal{D}(\mathbb{Q}_p^n), \|\cdot\|_\infty) \rightarrow C_0(\mathbb{Q}_p^n)$ is a bounded operator.

The proof of the Claim is as follows. Take $f \in \mathcal{D}(\mathbb{Q}_p^n)$ and $t > 0$, then

$$\begin{aligned} |T_t f(x)| &= \left| \int_{\mathbb{Q}_p^n} p_t(x, d^n y) f(y) \right| \leq \int_{\mathbb{Q}_p^n} p_t(x, d^n y) |f(y)| \\ &= \int_{\mathbb{Q}_p^n \setminus \{0\}} \tilde{Z}_t(y) |f(x - y)| d^n y \leq \|f\|_\infty \int_{\mathbb{Q}_p^n \setminus \{0\}} \tilde{Z}_t(y) d^n y \\ &= \|f\|_\infty p_t(0, \mathbb{Q}_p^n) = \|f\|_\infty, \end{aligned}$$

cf. **Claim 1** in the proof of Proposition 3, this shows that T_t is a linear bounded operator from $(\mathcal{D}(\mathbb{Q}_p^n), \|\cdot\|_\infty)$ into $L^\infty(\mathbb{Q}_p^n)$. We now show that $\lim_{x \rightarrow \infty} T_t f(x) = 0$. Take $f \in \mathcal{D}(\mathbb{Q}_p^n)$, with $\text{supp} f = E$ and $t > 0$, then

$$\begin{aligned} |T_t f(x)| &= \left| \int_{\mathbb{Q}_p^n} p_t(x, d^n y) f(y) \right| \leq \|f\|_\infty \int_E \tilde{Z}_t(x - y) d^n y \\ &\leq Ct \|f\|_\infty \int_E \|x - y\|_p^{-n} d^n y = Ct \|f\|_\infty \|x\|_p^{-n} \text{vol}(E), \end{aligned}$$

for $\|x\|_p$ big enough, cf. Proposition 2-(i). Finally, we show that $\lim_{x \rightarrow x_0} T_t f(x) = T_t f(x_0)$ for $t > 0$. This fact follows by using the Dominated Convergence Theorem, since

$$T_t f(x) = \int_{\mathbb{Q}_p^n \setminus \{0\}} \tilde{Z}_t(y) f(x - y) d^n y$$

and $\left| \tilde{Z}_t(y) f(x - y) \right| \leq \|f\|_\infty \tilde{Z}_t(y)$ with $\int_{\mathbb{Q}_p^n \setminus \{0\}} \tilde{Z}_t(y) d^n y = p_t(0, \mathbb{Q}_p^n) = 1$, cf. **Claim 1** in the proof of Proposition 3. \square

Remark 7. (i) We recall some results on Hunt, Lévy and Markov processes that we need to establish the main theorem of this section. All our processes have state space $(\mathbb{Q}_p^n, \mathcal{B}(\mathbb{Q}_p^n))$. A Hunt process is a Markov standard process which is quasi-left continuous on $[0, \infty)$, see [11, Definition 9.2 and the accompanying remarks].

(ii) Let $X = \{X_t\}_{t \geq 0}$ be a Hunt process with state space \mathbb{Q}_p^n and adjoined terminal state ∂ is a Lévy process on \mathbb{Q}_p^n if (1) $P^x(X_t \in E) = P^0(X_t + x \in E)$ for $t \geq 0$, $0, x \in \mathbb{Q}_p^n$ and E a Borel subset of \mathbb{Q}_p^n ; and (2) $P^0(X_t \in \mathbb{Q}_p^n) = 1$ for $t \geq 0$. Here $p_t(x, E) = P^x(X_t \in E)$.

(iii) The family of Borel probability measures $\{\mu_t, t \geq 0\}$ given by

$$(6.4) \quad \mu_t(E) = P^0(X_t \in E)$$

is a convolution semigroup such that

$$(6.5) \quad \mu_t \rightarrow \delta \text{ as } t \rightarrow 0^+,$$

where δ denotes the Dirac distribution. Conversely, it can be shown that for any convolution semigroup $\{\mu_t, t \geq 0\}$ satisfying (6.5), it is possible to construct a Lévy process X_t with state space \mathbb{Q}_p^n such that (6.4) is satisfied, see [20, Section 2] and [11, Exercise I-9-14].

Theorem 2. There exists a Lévy process $\mathfrak{X}(t, \omega)$, with state space $(\mathbb{Q}_p^n, \mathcal{B}(\mathbb{Q}_p^n))$ and transition function $p_t(x, \cdot)$.

Proof. We first show that there exists a Hunt process $\mathfrak{X}(t, \omega)$ with state space $(\mathbb{Q}_p^n, \mathcal{B}(\mathbb{Q}_p^n))$ and transition function $p_t(x, \cdot)$. This result follow from [11, Theorem 9.4] by Proposition 3, Lemma 6 and Remark 6-(i), (iii). On the other hand, from Remarks 7 and 6-(ii), it follows that the Hunt process constructed is a Lévy process. \square

7. FIRST PASSAGE TIME PROBLEM

Consider the following Cauchy problem:

$$(7.1) \quad \begin{cases} \frac{\partial u}{\partial t}(x, t) = Au(x, t), & t \in [0, \infty), x \in \mathbb{Q}_p^n \\ u(x, 0) = \Omega(\|x\|_p). \end{cases}$$

By Proposition 1,

$$(7.2) \quad u(x, t) = Z_t(x) * \Omega(\|x\|_p),$$

is a classical solution of (7.1). We now define

$$q_t(x, E) = \begin{cases} (u(\cdot, t) * 1_E)(x) & \text{for } t > 0 \text{ and } E \in \mathcal{B}(\mathbb{Q}_p^n) \\ 1_E(x) & \text{for } t = 0 \text{ and } E \in \mathcal{B}(\mathbb{Q}_p^n). \end{cases}$$

Since

$$\mathcal{D}(\mathbb{Q}_p^n) \rightarrow \mathbb{C}$$

$$\phi(x) \rightarrow \phi(x) * \Omega(\|x\|_p)$$

is linear continuous mapping, by the arguments given in the proof of Proposition 3, $q_t(x, E)$ is the transition function of a Markov process $\mathfrak{J}(t, \omega)$. Thus there exists a probability space $(\Upsilon, \mathcal{F}, P)$ and $\mathfrak{J}(t, \cdot) : (\Upsilon, \mathcal{F}, P) \rightarrow (\mathbb{Q}_p^n, \mathcal{B}(\mathbb{Q}_p^n), d^n x)$ is random variable for each $t \geq 0$. We notice that

$$\begin{aligned} P(\{\omega \in \Upsilon : \mathfrak{J}(0, \omega) \in \mathbb{Z}_p^n\}) &= q_0(0, \mathbb{Z}_p^n) = \Omega(\|x\|_p) * \Omega(\|x\|_p) |_{x=0} \\ &= \Omega(\|x\|_p) |_{x=0} = 1. \end{aligned}$$

In this section we study the following random variable.

Definition 4. The random variable $\tau_{\mathbb{Z}_p^n}(\omega) : \Upsilon \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$\inf\{t > 0; \mathfrak{J}(t, \omega) \in \mathbb{Z}_p^n \text{ and there exists } t' \text{ such that } 0 < t' < t \text{ and } \mathfrak{J}(t', \omega) \notin \mathbb{Z}_p^n\}$$

is called the first passage time of a path of the random process $\mathfrak{J}(t, \omega)$ entering the domain \mathbb{Z}_p^n .

Remark 8. We notice that the condition

$$(7.3) \quad P(\{\omega \in \Upsilon : \tau_{\mathbb{Z}_p^n}(\omega) < \infty\}) = 1$$

means that every path of $\mathfrak{J}(t, \omega)$ is sure to return to \mathbb{Z}_p^n . If (7.3) does not hold, then there exist paths of $\mathfrak{J}(t, \omega)$ that abandon \mathbb{Z}_p^n and never go back.

Lemma 7. The function $u(x, t) = Z_t(x) * \Omega(\|x\|_p)$, $t \geq 0$, is pointwise differentiable in t and its derivative is given by the formula

$$\frac{\partial u}{\partial t}(x, t) = \int_{\mathbb{Z}_p^n} \chi_p(-x, \xi) e^{(\hat{J}(\|\xi\|_p) - 1)t} [\hat{J}(\|\xi\|_p) - 1] d^n \xi, \text{ for } t \geq 0.$$

Proof. The formula is obtained by applying the Dominated Convergence Theorem. \square

Lemma 8. *The probability density function for a path of $\mathfrak{J}(t, \omega)$ to enter into \mathbb{Z}_p^n at the instant of time t , with the condition that $\mathfrak{J}(0, \omega) \in \mathbb{Z}_p^n$ is given by*

$$(7.4) \quad g(t) = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(\|y\|_p) u(y, t) d^n y.$$

Proof. The survival probability, by definition

$$S(t) := S_{\mathbb{Z}_p^n}(t) = \int_{\mathbb{Z}_p^n} u(x, t) d^n x,$$

is the probability that a path of $\mathfrak{J}(t, \omega)$ remains in \mathbb{Z}_p^n at the time t . Because there are no external or internal sources,

$$\begin{aligned} S'(t) &= \begin{array}{c} \text{Probability that a path of } \mathfrak{J}(t, \omega) \\ \text{goes back to } \mathbb{Z}_p^n \text{ at the time } t \end{array} - \begin{array}{c} \text{Probability that a path of } \mathfrak{J}(t, \omega) \\ \text{exists } \mathbb{Z}_p^n \text{ at the time } t \end{array} \\ &= g(t) - CS(t), \text{ with } 0 < C \leq 1. \end{aligned}$$

By using Lemma 7,

$$\begin{aligned} S'(t) &= \int_{\mathbb{Z}_p^n} \frac{\partial}{\partial t} u(x, t) d^n x = \int_{\mathbb{Z}_p^n} \left\{ \int_{\mathbb{Q}_p^n} J(\|y\|_p) u(x - y, t) d^n y - u(x, t) \right\} d^n x \\ &= \int_{\mathbb{Z}_p^n} \int_{\mathbb{Q}_p^n} J(\|y\|_p) \{u(x - y, t) - u(x, t)\} d^n y d^n x \\ &= \int_{\mathbb{Z}_p^n} \int_{\mathbb{Z}_p^n} J(\|y\|_p) \{u(x - y, t) - u(x, t)\} d^n y d^n x \\ &\quad + \int_{\mathbb{Z}_p^n} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(\|y\|_p) \{u(x - y, t) - u(x, t)\} d^n y d^n x. \end{aligned}$$

By Proposition 1, for $x, y \in \mathbb{Z}_p^n$,

$$\begin{aligned} u(x - y, t) &= \int_{\mathbb{Z}_p^n} \chi_p(-(x - y) \cdot \xi) e^{(\widehat{J}(\|\xi\|_p) - 1)t} d^n \xi = \int_{\mathbb{Z}_p^n} e^{(\widehat{J}(\|\xi\|_p) - 1)t} d^n \xi \\ &= u(x, t), \end{aligned}$$

i.e. $u(x - y, t) - u(x, t) \equiv 0$ for $x, y \in \mathbb{Z}_p^n$, consequently,

$$\begin{aligned} S'(t) &= \int_{\mathbb{Z}_p^n} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(\|y\|_p) (u(x - y, t) - u(x, t)) d^n y d^n x \\ &= \int_{\mathbb{Z}_p^n} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(\|y\|_p) u(x - y, t) d^n y d^n x - \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(\|y\|_p) d^n y \int_{\mathbb{Z}_p^n} u(x, t) d^n x \\ &= \int_{\mathbb{Z}_p^n} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(\|y\|_p) u(x - y, t) d^n y d^n x - CS(t), \end{aligned}$$

with $C := \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(\|y\|_p) d^n y \leq 1$, since J is of exponential type and $\int_{\mathbb{Q}_p^n} J(\|y\|_p) d^n y = 1$.

1. We notice that if $x \in \mathbb{Z}_p^n$ and $y \in \mathbb{Q}_p^n \setminus \mathbb{Z}_p^n$, then

$$\begin{aligned} u(x - y, t) &= \int_{\mathbb{Z}_p^n} \chi_p(-x \cdot \xi) \chi_p(y \cdot \xi) e^{(\widehat{J}(\|\xi\|_p) - 1)t} d^n \xi \\ &= \int_{\mathbb{Z}_p^n} \chi_p(y \cdot \xi) e^{(\widehat{J}(\|\xi\|_p) - 1)t} d^n \xi = \int_{\mathbb{Z}_p^n} \chi_p(-y \cdot \xi) e^{(\widehat{J}(\|\xi\|_p) - 1)t} d^n \xi = u(y, t), \end{aligned}$$

and consequently

$$\begin{aligned} S'(t) &= \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(\|y\|_p) u(y, t) d^n y - CS(t) \\ &= g(t) - CS(t) \text{ with } 0 < C \leq 1. \end{aligned}$$

□

Proposition 4. *The probability density function $f(t)$ of the random variable $\tau_{\mathbb{Z}_p^n}(\omega)$ satisfies the non-homogeneous Volterra equation of second kind*

$$(7.5) \quad g(t) = \int_0^\infty g(t - \tau) f(\tau) d\tau + f(t).$$

Proof. The result follow from Lemma 8 by using the argument given in the proof of Theorem 1 in [4]. □

Lemma 9. *The Laplace transform $G(s)$ of $g(t)$ is given by*

$$(7.6) \quad G(s) = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(\|y\|_p) \int_{\mathbb{Z}_p^n} \frac{\chi_p(-y \cdot \xi)}{s + (1 - \widehat{J}(\|\xi\|_p))} d^n \xi d^n y, \text{ for } \operatorname{Re}(s) > 0.$$

Proof. We first note that $e^{-st} J(\|y\|_p) e^{(\widehat{J}(\|\xi\|_p) - 1)t} \Omega(\|\xi\|_p) \in L^1((0, \infty) \times \mathbb{Q}_p^n \setminus \mathbb{Z}_p^n \times \mathbb{Q}_p^n, dt d^n \xi d^n y)$ for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$. The announced formula follows now from (7.4) and (7.2) by using Fubini's Theorem. □

Theorem 3. *If $-n < \gamma < 0$, then $P(\{\omega \in \Upsilon : \tau_{\mathbb{Z}_p^n}(\omega) < \infty\}) = 1$.*

Proof. By applying the Laplace transform to (7.5), we have

$$F(s) = \frac{G(s)}{1 + G(s)} = 1 - \frac{1}{1 + G(s)},$$

where $F(s)$ and $G(s)$ are the Laplace transforms of f and g , respectively. We understand $F(0) = \lim_{s \rightarrow 0} F(s)$ and $G(0) = \lim_{s \rightarrow 0} G(s)$. From $F(0) = \int_0^\infty f(t)dt = \frac{G(0)}{1+G(0)}$, it follows that if $G(0) = \infty$, then $\mathfrak{J}(t, \omega)$ is recurrent. Since $G(0) = \int_0^\infty g(t)dt$ is either a positive number or infinity, it is sufficient to show that $\lim_{s \rightarrow 0} G(s) = \infty$ for $s \in \mathbb{R}_+$. For $y \in \mathbb{Q}_p^n \setminus \mathbb{Z}_p^n$ with $\|y\|_p = p^i$, $i \in \mathbb{N} \setminus \{0\}$, we have

$$\begin{aligned} \int_{\mathbb{Z}_p^n} \frac{\chi_p(-y \cdot \xi)}{s+1-\widehat{J}(\|\xi\|_p)} d^n \xi &= \sum_{j=0}^{\infty} \frac{1}{s+1-\widehat{J}(p^{-j})} \int_{\|\xi\|_p=p^{-j}} \chi_p(-y \cdot \xi) d^n \xi \\ &= \sum_{j=i}^{\infty} \frac{p^{-nj}(1-p^{-n})}{s+1-\widehat{J}(p^{-j})} - \frac{p^{-ni}}{s+1-\widehat{J}(p^{1-i})}, \end{aligned}$$

and by (7.6),

$$\begin{aligned} G(s) &= \sum_{i=1}^{\infty} J(p^i) \int_{\|y\|_p=p^i} \left[\sum_{j=i}^{\infty} \frac{p^{-nj}(1-p^{-n})}{s+1-\widehat{J}(p^{-j})} - \frac{p^{-ni}}{s+1-\widehat{J}(p^{1-i})} \right] d^n y \\ &= \sum_{i=1}^{\infty} J(p^i) \left[\sum_{j=i}^{\infty} \frac{p^{-nj}(1-p^{-n})}{s+1-\widehat{J}(p^{-j})} - \frac{p^{-ni}}{s+1-\widehat{J}(p^{1-i})} \right] p^{ni}(1-p^{-n}) \\ &= (1-p^{-n}) \sum_{i=1}^{\infty} J(p^i) \left[(1-p^{-n}) \sum_{j=i}^{\infty} \frac{p^{n(i-j)}}{s+1-\widehat{J}(p^{-j})} - \frac{1}{s+1-\widehat{J}(p^{1-i})} \right]. \end{aligned}$$

Now, since $\lim_{j \rightarrow \infty} 1 - \widehat{J}(p^{-j}) = 0$, given any $s > 0$, there exists $j_0(s) \in \mathbb{N}$ such that $1 - \widehat{J}(p^{-j}) < s$ for $j > j_0(s)$. In addition, $s \rightarrow 0^+$ implies that $j_0(s) \rightarrow \infty$. By using these observations, we have

$$\begin{aligned} G(s) &\geq (1-p^{-n})J(p) \left[(1-p^{-n}) \sum_{j=1}^{\infty} \frac{p^{n(1-j)}}{s+1-\widehat{J}(p^{-j})} - \frac{1}{s+1-\widehat{J}(1)} \right] \\ &\geq (1-p^{-n})J(p) \left[\frac{p^n(1-p^{-n})}{2} \sum_{j=1}^{j_0(s)} \frac{p^{-nj}}{1-\widehat{J}(p^{-j})} - \frac{1}{s+1-\widehat{J}(1)} \right], \end{aligned}$$

notice that by Remark 2, $1 - \widehat{J}(1) > 0$. Therefore,

$$\begin{aligned} \lim_{s \rightarrow 0^+} G(s) &\geq (1-p^{-n})J(p) \left[\frac{p^n(1-p^{-n})}{2} \sum_{j=0}^{\infty} \frac{p^{-nj}}{1-\widehat{J}(p^{-j})} - \frac{1 + \frac{p^n(1-p^{-n})}{2}}{1-\widehat{J}(1)} \right] \\ &= (1-p^{-n})J(p) \left[\frac{p^n}{2} \int_{\mathbb{Z}_p^n} \frac{d^n \xi}{1-\widehat{J}(\|\xi\|_p)} - \frac{1 + \frac{p^n(1-p^{-n})}{2}}{1-\widehat{J}(1)} \right] = \infty, \end{aligned}$$

cf. Lemma 2. □

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